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A CONVERGENCE OF NONLINEAR CHAINS IN A ONE-DIMENSIONAL SETTING

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ABSTRACT: In one dimension, we look into unharmonic chain collisions with a hard wall. It is discovered that the coefficient of restitution (COR) has a minimum value at a specific velocity and is strongly impacted by the velocity at which chains meet. The link between COR and collision velocity is determined for low-velocity collisions using perturbation methods. The velocity dependence was found to be characterized by the exponent of the lowest unharmonic term of the interparticle potential energy.
Keywords: Convergence,Nonlinear,Dimensional,Unharmonic,Velocity

1. INTRODUCTION

Internal degrees of freedom cause some of the initial translational energy to be transformed into the internal energy of two bodies during collisions. This is the main source of energy dissipation. To characterize macroscopic qualities, the phenomenological quantity known as the coefficient of restitution (COR) is used.

$$\eta = \frac{K_r}{K_i},$$

It is usual to indicate the translational kinetic energy prior and after to the collision as Ki and Kr, respectively. Recently, collision investigations have mostly focused on determining γ using microscopic features.

Hertz's static theory of elastic contact served as the foundation for his concept of frictionless elastic entities meeting. According to the theory, in low-velocity collisions, the static theory governs the deformation of colliding bodies while ignoring vibration generation entirely. Because of this, the theory provides no particular information about energy dissipation. Plastic deformation is a potential technique for dissipating kinetic energy in collisions. The energy dissipation hypothesis is still viable in light of this. As a result, nonlinear effects play an important role in actual collisions. In this study, we simulate one-dimensional collisions between a rigid wall and a "nonlinear chain," with the interparticle interaction represented by a nonlinear force. We use perturbation methods to determine the COR's velocity dependency in low-velocity collisions. Our analysis shows that the COR for such collisions achieves its lowest point at a given velocity.

The current document is organized as follows. In the next part, we present a quick summary of the experiments undertaken by Basile and Dumont, as well as Sugiyama and Sasaki. The findings of our inquiry into the collision of nonlinear chains with a wall are described in Section 3. We finish with a summary of our findings.

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2. COLLISIONS OF ONE-DIMENSIONAL HARMONICCHAINS

Sugiyama and Sasaki's description of the collision of one-dimensional harmonic chains (model A) with a stiff wall is briefly examined first. Observe a sequence of n identical point particles, labeled j = 1, 2, • n. As depicted in Figure. A Hooke's spring arrangement connects each particle in the chain to the particles closest to it. The Hamiltonian for this system is given as

$$\mathcal{H} = \frac{m}{2}\sum_{j=1}^{n}\dot{x}_j^2 + \frac{1}{2}m\omega^2\sum_{j=1}^{n-1}(x_{i+1} - x_i - \ell)^2 + V_w, \quad (2)$$

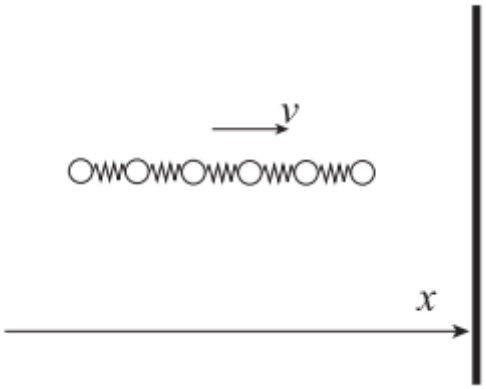

Figure 1 shows a diagram of a one-dimensional chain and a stiff wall.

In this context, "ℏ" represents the springs' natural length, whereas "xj" and "ɕxj" represent the position and velocity of the jth particle, respectively. The chain is considered to be homogeneous, with each spring having the same spring constant (k = mω2). The point of impact, Vw, is the stiff wall's hard-core potential at x = 0. The particle positional order xn −1 • • • x2 ≒ x1 ensures that only particle j = 1 at the end of the chain interacts with the wall. In the following explanation, we will assume that the wall is hard enough to cause the particle x1's velocity to reverse, indicated as −xɕ 1 → xн 1.

The residual energy remaining after a collision is regarded to have "dissipated," despite the Hamiltonian's lack of a dissipation component. As a result, the COR is calculated as

$$\eta = 1 - \frac{E_{vib.}}{E},$$

in which Evib. is the vibration energy that remains after the collision, and E is the overall energy. In the absence of Vw, the equation of motion is expressed as

$$\ddot{\mathbf{x}} = -\omega^2(A\mathbf{x} + \mathbf{b}),$$

A is a n × n matrix, with x and b as (–™, 0, • •, 0, ™) and (x1, x2, • •, xn), respectively.

$$A = \begin{pmatrix} 1 & -1 & 0 & & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 \; -1 & 1 \end{pmatrix}.$$

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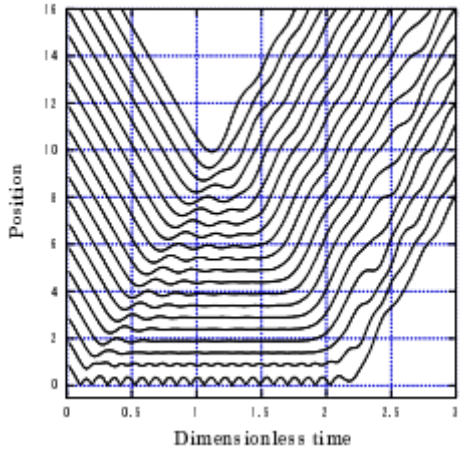
The chain's internal vibration can be represented by N noninteracting fundamental modes in a principal-axis coordinate system with A diagonal. Vw causes the chain and wall to clash, as explained in the following section. It is assumed that the particle j = 1 collides with the wall at time t1, t2,..., tf(n). Equal-energy elliptical orbits can be recognized by their fundamental modes in phase space. The orbits exhibit discontinuities at the time constants t = t1, t2, • •, and tf(n).

The initial condition is chosen for further numerical simulations.

$$\begin{cases} x_j &= \ell\, j, \\ \dot{x}_j &= -v_0 \quad (j = 1, 2, \cdots, n). \end{cases}$$

Prior to contact, the chain has either a negative temperature or no internal vibration. Figure 2 depicts the collision between the wall and the chain with n=19. Each line indicates a particle's trajectory. The units on the time axis of the graph are τ = (n-1)$\hbar$/cl. τ denotes the period of longitudinal sound wave transmission from one end of the chain to the other. Consider the instant tc, which represents the occurrence of the collision, as "contact time." This means that the contact time is approximately equal to 2τ.



The trajectory of particles in the n = 20 colliding chain is shown in FIG. 2. Decrease in units when cl = τ = 1.

with a huge number. The collision velocity of the chain has no effect on COR or contact duration since time intervals △tq = tq+1 → tq stay constant for all values of q and have similar orbital morphologies. As a result, Ej/E, which indicates the ratio of total energy to the j-th basic mode energy, is unaffected by beginning velocity. In addition, the energy ratio COR is unaffected by beginning velocity. To proceed, we will set $\hbar$, m, and k to unity and use the reduced unit cl = τ = 1.

Model A's coefficient of restitution (γ) is solely determined by the value of n. Dumont and Basile found an approximation for the symbol η. To simplify the problem, we assume that △tq and vq, the velocities previous to the qth collision, are constant for all values of q. Numerical calculations indicate that these estimations are feasible. Assuming this, the vibration energy of the j-th mode following the collision technique can be represented as:

$$E_j = \frac{4m}{n} v^2 \cos^2\left(\frac{\pi j}{2n}\right) \frac{\sin^2\{\omega_j f(n) \triangle t/2\}}{\sin^2\{\omega_j \triangle t/2\}},$$

where ωj is the frequency of the j-th mode and equals sin(πj/n). Numerical simulations are

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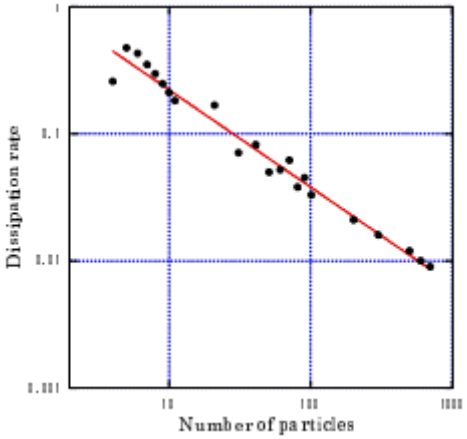
used to estimate △t, f(n), and v . However, it is also possible to compute these numbers using n = 2 to resolve the chain's collision:

$$\begin{aligned} \Delta t &= \frac{\pi}{\sqrt{2}\omega} = 2.22/\omega \quad (2.31/\omega), \\ f(n) &\simeq \frac{2\sqrt{2}}{\pi} n = 0.90n \quad (0.867n), \\ v &= \frac{\pi}{2\sqrt{2}} v_0 = 1.11 v_0 \quad (1.15 v_0), \end{aligned}$$

The values in brackets were obtained by numerically simulating a chain with n = 500. Equations (8), (9) and (10) provide an approximation for determining η's value.



The figure depicts a log-logarithmic relationship between the dissipation rate and the particle count. 3. The slope of the solid line is -2/3. Decrease in units when cl = τ = 1.

We generalize the dispersion relation to obtain the asymptotic behavior of 1−η for large n values.

$$\omega_j = \sqrt{k/m}\frac{\pi j}{n}\left(1 - \frac{\pi^2 j^2}{24n^2} + \cdots\right)$$

To derive η = 1, just replace the preceding order of Equation (11) with Eq. (7). Taking into consideration the second order, we get.

$$1 - \eta \sim 0.652\, n^{-\frac{2}{3}}, \qquad 1 - \eta \sim 0.652\, n^{-\frac{2}{3}},$$

Within the range of n to infinity. There is a strong association between this and numerical results. Figure illustrates the dissipation rate 1 − η versus n. 3.

3. UNHARMONIC CHAINS

Initial velocity v0 had no obvious effect on the COR or contact time of harmonic chains. We apply nonlinearity to the chain's springs (model B). To preserve uniformity, we begin by looking at a velocity scale defined by the spring's nonlinearity. The potential energy of the spring, denoted as U(x), is assumed to be U(1) = 0, provided that the spring's inherent length equals unity. The answer can be used to find the amplitude (x ∏) where the harmonic term and the sum of the other components in the Taylor expansion are equal.

$$\frac{1}{2}\frac{d^2U(x)}{dx^2}\bigg|_{x=1}(x^* - 1)^2 = U(x) - \frac{1}{2}\frac{d^2U(x)}{dx^2}\bigg|_{x=1}(x^* - 1)^2.$$

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The velocity, represented as x ∏, is

$$v^* = \sqrt{2U(x^*)}.$$

Next, we investigate the velocity dependence of collisions on the v ∗ scale.

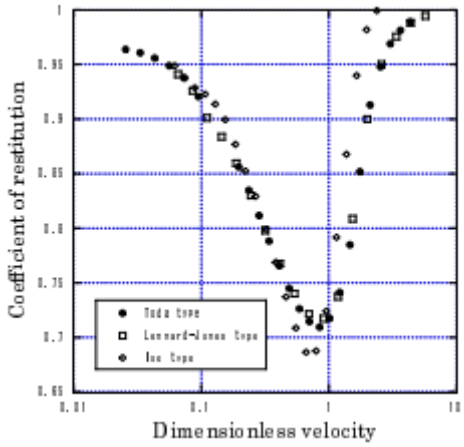

Figure 4 depicts the association between impact velocity and coefficient of restitution for chains of the log, Toda, and Lennard-Jones kinds. Each chain consists of one hundred pieces. The following three probable categories will be selected, and the simulation outcomes will be compared.

(a) Lennard-Jones potential

$$U_{LJ}(x) = \frac{1}{72}\left\{\left(\frac{1}{x}\right)^{12} - 2\left(\frac{1}{x}\right)^{6}\right\}$$

(b) Toda potential

$$U_{toda}(x) = \frac{a}{b}e^{-b(x-1)} + a(x-1)$$

(c) Log-type potential

$$U_{log}(x) = x - \log(x)$$

The Toda potential has both ab and b values of one. and 10. Each potential has a single minimum at x = 1, and the functions' shapes are comparable. In contrast, the expanding behaviors of the repulsive forces emanating from the three potentials vary significantly across a very short distance. As shown in Figure 4, we plot the COR against the beginning velocity for each occurrence, as estimated by numerical simulations. At low velocity, COR approaches the value discovered for the harmonic chain. The minimum value of the COR occurs at v = 1, and it decreases as the initial velocity increases. The COR is observed to be on a nearly identical trajectory for all three probable groups.

Figure 5 illustrates the relationship between γ and n for various beginning velocity levels. In a collision at extremely low velocities, the dissipation rate 1−η approaches zero in the thermodynamic limit, as predicted by model A. Even when v0 = 1 is the thermodynamic limit, the COR remains less than unity. Currently, it does not approach unity. As a result, the lack of dissipation in model A can be attributed to the potential U(x)'s nonlinearity. Using a perturbative theory-based technique, we regard model B's collision to be a modest starting event.

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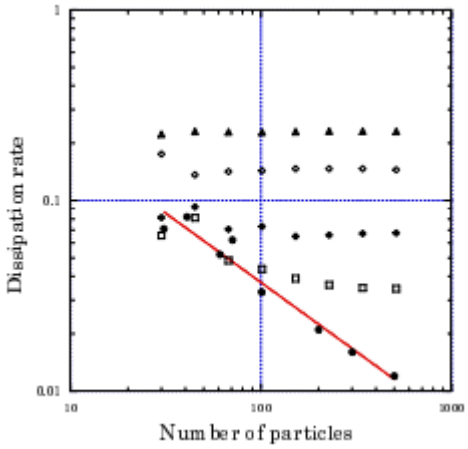

Figure 5 depicts the relationship between dissipation rate and particle count in a chain collision of the Lennard-Jones type. Each panel has the same initial velocities: v0 = 0.780 (N), 0.390 (□), 0.160 (), 0.078 (), and 0.008 (•).

v0 velocity greater than or equal to one. During collision processes, the particle j = 1 transfers vibrational force to the next particle. The force exerted on the chain will be indicated as the external force F(t). The characteristic frequency of this force is indicated by ωext = 2π/△t ≃ 2 - 2ω. The frequency exceeds the total of the chain's basic modes. As a result, the force fails to activate any fundamental mode. Particle j = 1 has the maximum amplitude during a collision because the amplitude of each particle's vibration is rapidly and gradually dampened as it enters the chain. It is assumed that when the initial velocity increases, t decreases. In Eq., we simply study the fluctuation of △t in relation to the initial velocity v0. (7) is the initial estimate. To estimate the velocity dependence of ~~A~~t, analyze the collision of a Lennard-Jones chain with n=2. In this case, particle j = 1 makes two hits against the wall. The Hamiltonian signifies

$$\mathcal{H} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + U_{LJ}(x_2 - x_1 - 1) + V_w.$$

Define x as x2 → x1 → $\hbar$, and xg as (x2 + x1)/2. We use Eq. (6) With relation to the original conditions. Particles x1, x, and xg, as well as their temporal derivatives, are rapidly altered after their initial impact.

$$\begin{cases} x &= 0 \\ \dot{x} &= -2v_0 \end{cases}, \qquad \begin{cases} x_g &= \ell/2 \\ \dot{x}_g &= 0 \end{cases}.$$

Incorporating the second order of ULJ and using the beginning conditions of Eq. Computing the equation of motion yields the interval △t between particle x1's initial and second collisions. (14) in the theory of first-order perturbation.

$$\triangle t \simeq \frac{\pi}{\sqrt{2}}\left(1 - \alpha\frac{v_0}{v^*}\right),$$

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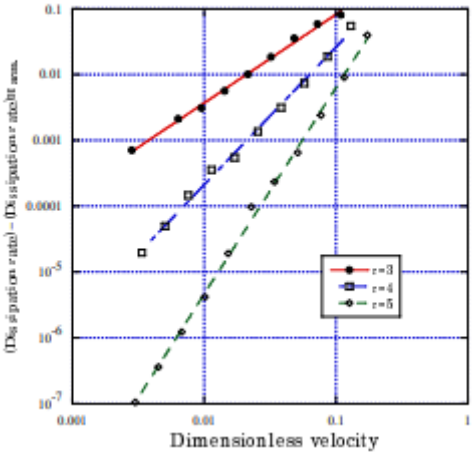

Dissipation rates are rising (see Figure 6). We plot the dissipation rate of model B as a function of r for r values of 3, 4, and 5, after subtracting model A's dissipation rate. The fitted lines have slopes of 1.34, 2.09, and 3.14.

By using α = 0.808. After inserting this into the equation. (7) We obtain.

In the range n → ���, the linear form ωj is interchangeable.

$$\omega_j = \omega \frac{\pi j}{n}.$$

When v/cl > 1, the equation can be tackled. (16) as

$$E_j \simeq \frac{32}{n\pi^2}\left(\frac{n}{j}\right)^2 \sin^2\left[\alpha v \pi n \left(\frac{j}{n}\right)\right].$$

Substituting this into Eq. (3), we have the dissipation rate

$$\begin{aligned} 1-\eta = \sum_{j=1}^{n} \frac{E_j}{E} &\simeq \frac{64}{n\pi^4}\int_0^1 dx \frac{1}{x^2}\sin^2(\alpha v n x) \\ &\simeq \frac{64}{\pi^3}\alpha v \int_0^\infty dx \frac{\sin^2(\alpha x)}{x^2} \\ &= Cv, \end{aligned}$$

By entering this value into Equation (3), one can calculate the dissipation rate.

In the case of C, 32π2 α equals 2.61. This means that the dissipation rate increases as p approaches 1 according to the power equation v p, when p = 1. This finding for low-velocity collisions is consistent with computer models. In contrast, the aforementioned result deviates from the constant C, as evidenced by our numerical simulation, which yields C = 0.66.

The exponent of the lowest non-harmonic term in the interparticle potential energy has a direct bearing on this finding. For the Lennard-Jones potential, r equals three. In the case of alternate chain types, the exponent remains constant. Let's talk about more common cases. Assume the interparticle potential energy can be extended toward its equilibrium position.

$$U(x) = \frac{1}{2}x^2 + cx^r + \text{ higher order},$$

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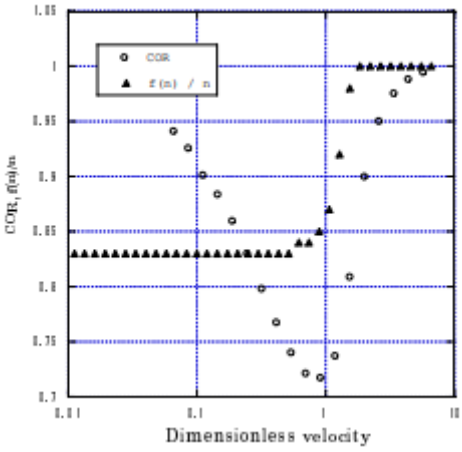

Figure 7 depicts the Lennard-Jones type chain's velocity as a function of the f(n) to n ratio and COR. Both climb at $^{y}v \approx 1$. represented by the positive (negative) constant c.

R is an odd number (even). In this scenario, contact time reduces as v r −2 0. Thus, the rate of dissipation accelerates.

$$1 - \eta \propto \left(\frac{v}{v^*}\right)^{r-2}.$$

By using the potential energy equation. Figure 19 displays numerical COR findings for $v \geq 1$ to demonstrate particle interaction in model B. 6. The results agree with Equation (20). The δ function explains particle interactions in a chain of rigid spheres. The collision of the two spheres is effectively an exchange of momentum. It is possible to obtain an exact solution for the overall dynamics of a chain colliding with a rigid wall. When the sphere j = 1 contacts the wall n times, there is no residual internal vibration; in this case, the COR is exactly one. Model B's high-velocity collision limit causes particles to behave firmly as spheres due to the short-range behavior of ULJ, Utoda, and Ulog as hard core potentials. Figure 7 depicts the COR of a Lennard-Jones-type chain as well as the relationship between f(n)/n and collision velocity. When v exceeds one, f(n)/n approaches unity. This implies that particles behave like rigid spheres in high-velocity collisions. As a result, COR increases during the high velocity regime. This particular attribute is deemed impractical because it is unique to the one-dimensional model. In actual systems, plastic deformation is a major factor in high-velocity collisions.

4. SUMMARY

To better understand the energy dissipation process, we created a one-dimensional fundamental microscopic model of bodies colliding. The Lennard-Jones, Toda, and Logarithmic potentials are used to represent particle interactions. It was noticed that the COR is dependent on the initial velocity and attains its minimum at $v/v* \leq 1$. The potential form does not exert any influence on these actions. In low-velocity collisions, the link between energy dissipation rate and collision velocity is determined via pinchation methods.

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